Cartesian closedness in convenient setting

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1 Remainder

Definition 1. convenient vector space

Definition 2. c^∞-topology

Definition 3. direct (inductive) limit space Let $(I, \prec)$ be a directed set. Let $\{A_i : i \in I\}$ be a family of objects indexed by I and $f_{ij} : A_i \to A_j$ be a homomorphism for all $i \prec j$ with the following properties:

1. f_{ii} is the identity of A_i, and
2. $f_{ik} = f_{jk} \circ f_{ij}$ for all $i \prec j \prec k$.

The pair (A_i, f_{ij}) is called a direct system over I. The underlying set of the direct limit, $\varinjlim A_i$, of the direct system (A_i, f_{ij}) is defined as the disjoint union of the A_i's modulo an equivalence relation $\sim$:

$$\varinjlim A_i = \coprod_i A_i \Big/ \sim . \tag{1}$$

For $x_i \in A_i$ and $x_j \in A_j$, $x_i \sim x_j$ if there is some $k \in I$ such that $f_{ik}(x_i) = f_{jk}(x_j)$. One has canonical morphisms $\phi_i : A_i \to A$ sending each element to its equivalence class.

Consider a case where objects are lcvs (A_i, τ_i). If the connecting mappings are continuous and $\varinjlim A_i$ is closed in $\coprod_i A_i$ we can equip $\varinjlim A_i$ with the induced Hausdorff topology and we speak of **inductive limit topology**.

The inductive limit is **strict** if $A_i \subset A_j$ for $i \prec j$ and the topology induced by τ_j on the subspace A_i of A_j is equal to τ_i.

Definition 4. inverse (projective) limit space Let $(I, \prec)$ be a directed set, Let $\{A_i : i \in I\}$ be a family of objects indexed by I and $f_{ij} : A_j \to A_i$ be a homomorphism for all $i \prec j$ with the following properties:

1. f_{ii} is the identity in A_i,
2. $f_{ik} = f_{ij} \circ f_{jk}$ for all $i \prec j \prec k$.

The pair (A_i, f_{ij}) is called an inverse system over I. We define the inverse limit $\varprojlim A_i$ of the inverse system (A_i, f_{ij}) as a particular subspace of the direct product of the A_i's:

$$\varprojlim A_i = \Big\{ (a_i) \in \prod_{i \in I} A_i \;\Big|\; a_i = f_{ij}(a_j) \text{ for all } i \leq j \Big\}. \tag{2}$$

The inverse limit is equipped with natural projections $\pi_i : A \to A_i$ which pick out the i-th component of the direct product.

Definition 5. Let $(X, \mathcal{B})$ be a bornological set. Define $\ell^\infty(X, F)$ to be the space of all functions $f : X \to F$, which are bounded on all $B \in \mathcal{B}$, supplied with the topology of uniform convergence on the sets in $\mathcal{B}$.

The following constructions preserve c^∞-completeness:

- inverse limits
- direct sums
- strict inductive limits of sequences of closed embeddings
- formation of $\ell^\infty(X, \bullet)$

2 Exponential law for $U = V = E = \mathbb{R}$

Recall from calculus:

Theorem 6. *Let $f : \mathbb{R}^2 \to \mathbb{R}$ be an arbitrary mapping. Then all iterated partial derivatives exist and are locally bounded if and only if the associated mapping $f^\vee : \mathbb{R} \to \mathcal{C}^\infty(\mathbb{R}, \mathbb{R})$ exists as a smooth curve, where $\mathcal{C}^\infty(\mathbb{R}, \mathbb{R})$ is considered as the Fréchet space with the topology of uniform convergence of each derivative on compact sets. Furthermore, we have $(\partial_1 f)^\vee = d(f^\vee)$ and $(\partial_2 f)^\vee = d \circ f^\vee$.*

Theorem 7. *[Boman, 1967] For a mapping $f : \mathbb{R}^2 \to \mathbb{R}$ the following assertions are equivalent:*

1. *All iterated partial derivatives exist and are continuous.*
2. *All iterated partial derivatives exist and are locally bounded.*
3. *For $v \in \mathbb{R}^2$ the iterated directional derivatives:*
$$d_v^n f(x) := \left(\frac{\partial}{\partial t}\right)^n \Big|_{t=0} (f(x + tv)) \tag{3}$$
exist and are locally bounded with respect to x.
4. *For all smooth curves $c : \mathbb{R} \to \mathbb{R}^2$ the composite $f \circ c$ is smooth.*

Lemma 8. *Let $f_\epsilon \to f$ in $C(\mathbb{R}^2, \mathbb{R})$ and $d_v f_\epsilon \to f_v$ in $C(\mathbb{R}^2, \mathbb{R})$. Then $d_v f$ exists and equals f_v .*

Proof. Show that for fixed $x, v \in \mathbb{R}^2$ the curve:
$$c : t \mapsto \begin{cases} \frac{f(x+tv)-f(x)}{t} & \text{for } t \neq 0 \\ f_v(x) & \text{otherwise} \end{cases} \tag{4}$$
is continuous from $\mathbb{R} \to \mathbb{R}$. The corresponding curve c_ϵ for f_ϵ can be rewritten as $c_\epsilon(t) = \int_0^1 d_v f_\epsilon(x + \tau t v) d\tau$, which converges by assumption uniformly for t in compact sets to the continuous curve $t \mapsto \int_0^1 f_v(x + \tau t v) d\tau$. Pointwise it converges to $c(t)$, hence c is continuous. $\square$

3 Exponential law for $U = V = \mathbb{R}$, E lcvs

Definition 9. We define $\mathcal{C}^\infty(\mathbb{R}, E)$ to be the locally convex vector space of all smooth curves into E, with the pointwise vector operations, and with the topology of **uniform convergence on compact sets of each derivative separately**. This is the initial topology with respect to the linear mappings $\mathcal{C}^\infty(\mathbb{R}, E) \xrightarrow{d^k} \mathcal{C}^\infty(\mathbb{R}, E) \to \ell^\infty(K, E)$, where $k \in \mathbb{N}$, K runs through all compact subsets of $\mathbb{R}$.

Lemma 10. *A space E is c^∞-complete if and only if $\mathcal{C}^\infty(\mathbb{R}, E)$ is.*

Proof. $(\Rightarrow)$. The mapping $c \mapsto (c^{(n)})_{n\in\mathbb{N}}$ is by definition an embedding of $\mathcal{C}^\infty(\mathbb{R}, E)$ into the c^∞-complete product $\prod_{n\in\mathbb{N}} \ell^\infty(\mathbb{R}, E)$. It's image is a closed subspace, since lemma 8 can be easily generalized to curves $c : \mathbb{R} \to E$.
$(\Leftarrow)$. Consider the continuous linear mapping $const. : E \to \mathcal{C}^\infty(\mathbb{R}, E)$ given by $x \mapsto (t \mapsto x)$. It has as continuous left inverse the evaluation at any point (e.g. $ev_0 : \mathcal{C}^\infty(\mathbb{R}, E) \to E$, $c \mapsto c(0)$). Hence, E can be identified with the closed subspace of $\mathcal{C}^\infty(\mathbb{R}, E)$ given by the constant curves, and is thereby itself c^∞-complete. □

Lemma 11. *A curve into a c^∞-closed subspace of a space is smooth if and only if it is smooth into the total space. In particular, a curve is smooth into a projective limit if and only if all its components are smooth.*

Proof. Since the derivative of a smooth curve is the Mackey limit of the difference quotient, the c^∞-closedness implies that this limit belongs to the subspace. Thus, we deduce inductively that all derivatives belong to the subspace, and hence the curve is smooth into the subspace. The result on projective limits now follows, since obviously a curve is smooth into a product, if all its components are smooth. □

Remark 1. *The* **bornology** *on function spaces can be tested with the linear functionals on the range space.*

Lemma 12. *The family $\{\ell_* : \mathcal{C}^\infty(\mathbb{R}, E) \to \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}) : \ell \in E^*\}$ generates the bornology of $\mathcal{C}^\infty(\mathbb{R}, E)$. This also holds for E^* replaced by E' .*

Proof. A set $B \subseteq \mathcal{C}^\infty(\mathbb{R}, E)$ is bounded if and only if the sets $\{d^n c(t) : t \in I, c \in B\}$ are bounded in E for all $n \in \mathbb{N}$ and compact subsets $I \subset \mathbb{R}$. This is furthermore equivalent to the condition that the set $\{\ell(d^n c(t)) = d^n(\ell \circ c)(t) : t \in I, c \in B\}$ is bounded in $\mathbb{R}$ for all $\ell \in E^*$, $n \in \mathbb{N}$, and compact subsets $I \subset \mathbb{R}$ and in turn equivalent to: $\{\ell \circ c : c \in B\}$ is bounded in $\mathcal{C}^\infty(\mathbb{R}, \mathbb{R})$.

For E^* replaced by $E' \supseteq E^*$ the statement holds, since ℓ is bounded for all $\ell \in E'$ by the explicit description of the bounded sets. □

Theorem 13. *For a mapping $f : \mathbb{R}^2 \to E$ into a locally convex space (which need not be c^∞-complete) the following assertions are equivalent:*

1. *f is smooth along smooth curves.*
2. *All iterated directional derivatives $d_v^p f$ exist and are locally bounded.*
3. *All iterated partial derivatives $\partial_\alpha f$ exist and are locally bounded.*

4. $f^\vee : \mathbb{R} \to \mathcal{C}^\infty(\mathbb{R}, E)$ *exists as a smooth curve.*

Proof. We shall consider two cases:

1. E is c^∞-complete. Each of the statements 1-4 is valid if and only if the corresponding statement for $\ell \circ f$ is valid for all $\ell \in E^*$. Remark to 4: In fact, $f^\vee(t) \in \mathcal{C}^\infty(\mathbb{R}, E)$ if and only if $\ell_*(f^\vee(t)) = (\ell \circ f)^\vee(t) \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R})$ for all $\ell \in E^*$ by 1. Since $\mathcal{C}^\infty(\mathbb{R}, E)$ is c^∞-complete, its bornologically isomorphic image in $\prod_{\ell \in E^*} \mathcal{C}^\infty(\mathbb{R}, \mathbb{R})$ is c^∞-closed. So $f^\vee : \mathbb{R} \to \mathcal{C}^\infty(\mathbb{R}, E)$ is smooth if and only if $\ell_* \circ f^\vee = (\ell \circ f)^\vee : \mathbb{R} \to \mathcal{C}^\infty(\mathbb{R}, \mathbb{R})$ is smooth for all $\ell \in E^*$. So the proof is reduced to the scalar case, which was proved in theorems 6 and 7.

2. The general case. For the existence of certain derivatives we know that it is enough that we have some candidate in the space, which is the corresponding derivative of the map considered as map into the c^∞-completion (or even some larger space). Since the derivatives required in (1-4) depend linearly on each other, this is true.

$\square$

4 Exponential law for general lcvs

Definition 14. A mapping $f : E \supseteq U \to F$ defined on a c^∞-open subset U is called smooth (or $\mathcal{C}^\infty$) if it maps smooth curves in U to smooth curves in F. By $\mathcal{C}^\infty(U, F)$ we shall denote the locally convex space of all smooth mappings $U \to F$ with pointwise linear structure and the initial topology with respect to all mappings $c^* : \mathcal{C}^\infty(U, F) \to \mathcal{C}^\infty(\mathbb{R}, F)$ for $c \in \mathcal{C}^\infty(\mathbb{R}, U)$.

Remark 2. *For $U = E = \mathbb{R}$ this coincides with our old definition.*

Lemma 15. *The space $\mathcal{C}^\infty(U, F)$ is the (inverse) limit of spaces $\mathcal{C}^\infty(\mathbb{R}, F)$, one for each $c \in \mathcal{C}^\infty(\mathbb{R}, U)$, where the connecting mappings are pullbacks g^* along reparameterizations $g \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R})$. Note that this limit is the closed linear subspace in the product $\prod\limits_{c \in \mathcal{C}^\infty(\mathbb{R}, U)} \mathcal{C}^\infty(\mathbb{R}, F)$ consisting of all (f_c) with $f_{c \circ g} = f_c \circ g$ for all c and all $g \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R})$.*

Proof. The mappings $c^* : \mathcal{C}^\infty(U, F) \to \mathcal{C}^\infty(\mathbb{R}, F)$ define a continuous linear embedding $\mathcal{C}^\infty(U, F) \to \lim_c\{\mathcal{C}^\infty(\mathbb{R}, F) \xrightarrow{g^*} \mathcal{C}^\infty(\mathbb{R}, F)\}$, since $c^*(f) \circ g = f \circ c \circ g = (c \circ g)^*(f)$. It is surjective since for any $(f_c) \in \lim_c \mathcal{C}^\infty(\mathbb{R}, F)$ one has $f_c = f \circ c$ where f is defined as $x \mapsto f_{const_x}(0)$. $\square$

Theorem 16. *Let $U_i \subseteq E_i$ be c^∞-open subsets in locally convex spaces, which need not be c^∞-complete. Then a mapping $f : U_1 \times U_2 \to F$ is smooth if and only if the canonically associated mapping $f^\vee : U_1 \to \mathcal{C}^\infty(U_2, F)$ exists and is smooth.*

Proof. We have the following implications:

$$
\begin{aligned}
& f^\vee : U_1 \to \mathcal{C}^\infty(U_2, F) \text{ is smooth} \\
\Leftrightarrow\ & f^\vee \circ c_1 : \mathbb{R} \to \mathcal{C}^\infty(U_2, F) \text{ is smooth for all smooth curves } c_1 \text{ in } U_1, \text{ by definition 14.} \\
\Leftrightarrow\ & c_2^* \circ f^\vee \circ c_1 : \mathbb{R} \to \mathcal{C}^\infty(\mathbb{R}, F) \text{ is smooth } \forall \text{ smooth } c_i \text{ in } U_i, \text{ by 14 and 11} \\
\Leftrightarrow\ & f \circ (c_1 \times c_2) = (c_2^* \circ f^\vee \circ c_1)^\wedge : \mathbb{R}^2 \to F \text{ is smooth for all smooth } c_i \text{ in } U_i, \text{ by 13} \\
\Leftrightarrow\ & f : U_1 \times U_2 \to F \text{ is smooth.}
\end{aligned}
$$

Remark to the last step: each curve into $U_1 \times U_2$ is of the form $(c_1, c_2) = (c_1 \times c_2) \circ \Delta$, where Δ is the diagonal mapping. Conversely, $f \circ (c_1 \times c_2) : \mathbb{R}^2 \to F$ is smooth for all smooth curves c_i in U_i , since the product and the composite of smooth mappings is smooth by 14 (and by 7). $\square$

Corollary 17. *Let $E, F, G, \ldots$ be locally convex spaces, and let U, V be c^∞-open subsets of such. Then the following canonical mappings are smooth:*

1. $ev : \mathcal{C}^\infty(U, F) \times U \to F$, $(f, x) \mapsto f(x)$
2. $ins : E \to \mathcal{C}^\infty(F, E \times F)$, $x \mapsto (y \mapsto (x, y))$
3. $(\bullet)^\wedge : \mathcal{C}^\infty(U, \mathcal{C}^\infty(V, G)) \to \mathcal{C}^\infty(U \times V, G)$
4. $(\bullet)^\vee : \mathcal{C}^\infty(U \times V, G) \to \mathcal{C}^\infty(U, \mathcal{C}^\infty(V, G))$
5. $comp : \mathcal{C}^\infty(F, G) \times \mathcal{C}^\infty(U, F) \to \mathcal{C}^\infty(U, G)$, $(f, g) \mapsto f \circ g$
6. $\mathcal{C}^\infty(\bullet, \bullet) : \mathcal{C}^\infty(E_2, E_1) \times \mathcal{C}^\infty(F_1, F_2) \to \mathcal{C}^\infty(\mathcal{C}^\infty(E_1, F_1), \mathcal{C}^\infty(E_2, F_2))$,
   $(f, g) \mapsto (h \mapsto g \circ h \circ f)$
7. $\prod : \prod \mathcal{C}^\infty(E_i, F_i) \to \mathcal{C}^\infty(\prod E_i, \prod F_i)$, *for any index set.*

Corollary 18 (Boman, 1967)**.** *The smooth mappings on open subsets of R^n in the sense of definition 14 are exactly the usual smooth mappings.*

Proposition 19. *Let $f : E \times \mathbb{R} \supseteq U \to F$ be smooth with c^∞-open $U \subseteq E \times \mathbb{R}$. Then $x \mapsto \int_0^1 f(x, t)dt$ is smooth on the c^∞-open set $U_0 := \{x \in E : \{x\} \times [0, 1] \subseteq U\}$ with values in the completion $\widehat{F}$ and $d_v f_0(x) = \int_0^1 d_v(f(\bullet, t))(x)dt$ for all $x \in U_0$ and $v \in E$.*

Definition 20. By $L(E, F)$ we denote the space of all bounded (equivalently smooth) linear mappings from E to F . It is a closed linear subspace of $\mathcal{C}^\infty(E, F)$ since f is linear if and only if for all $x, y \in E$ and $\lambda \in \mathbb{R}$ we have $(ev_x + \lambda ev_y ev_{x+\lambda y})f = 0$. We equip it with this topology and linear structure.

Theorem 21 (Chain rule)**.** *Let E and F be locally convex spaces, and let $U \subseteq E$ be c^∞-open. Then the differentiation operator:*

$$d \quad : \quad \mathcal{C}^\infty(U, F) \to \mathcal{C}^\infty(U, L(E, F)), \tag{5}$$

$$df(x)v \quad := \quad \lim_{t \to 0} \frac{f(x + tv) - f(x)}{t}, \tag{6}$$

exists, is linear and bounded (smooth). Also the chain rule holds:

$$d(f \circ g)(x)v = df(g(x))dg(x)v. \tag{7}$$

References

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[3] Bourbaki, N.; *Topological vector spaces*, ch. 1-5, Springer-Verlag Berlin, Heidelberg, New York 2003