Chapter 1

Bornology

1.1 Definitions

1.1.1

A **bornology** on a set X is a family B of subsets of X satisfiying the following axioms:

B is covering of X
B is hereditary under inclusion
B is stable under finite union

(X, B) is called a bornological set, and the elements of B are called bounded subsets of X.

A **base of a bornology** B is a subfamily B_0 of B, such that each element of B is a subset of an element of B_0.

1.1.2

Let E be a VS over K.
A bornology B on E is called a **vector bornology** on E if:

B is stable under vector addition
B is stable under homothetic transformations
B is stable under formation of circled hulls

(E,B) is called a **bornological vector space**.

1.1.3

A vector bornology B is called a **convex vector bornology** if it is stable under formation of convex hulls.

(E,B) is called a **convex bornological (vector) space**.

1.1.4

A **separated bornological vector space** (E,B) is a bornological VS, for which $\{0\}$ is the only bounded subVS of E.

1.2 Bounded Linear Maps

A map u between two bornological set X, Y is called a **bounded map**, if the image of each bounded subset of X is a bounded subset of Y.

A bornology B_1 on a set X is called a **finer** bornology than a bornology B_2 on X, if $id : (E, B_1) \to (E, B_2)$ is bounded.

A bijection u between two bornological sets X, Y is called a **bornological isomorphism** if u and u^{-1} are bounded.

1.2.1

A **bounded linear functional** on a bornological VS E over K is a bounded linear map of E into K, endowed with the bornology defined by the absolute value.

1.3 Fundamental Examples of Bornologies

Example 1:
Let K be a field with an absolute value. The family of subsets which are boundeed in respect to the absolute value is a convex bornology on K. It is called the **canonical bornology** of K.

Example 2:
Let E be a VS over K and let p be a seminorm on E. The collection of subsets A of E for which $p(A)$ is bounded in K is called the **canonical bornology of the seminormed space** (E, p). Note that this bornology is separated iff p is a norm.

Example 3:
Let $\Gamma = \{p_i\}$ be a collection of seminorms on E. Then the collection of subsets A of E for which $p_i(A)$ is boundede in K is a bornology on E. It is calles the **bornology defined by** Γ. It is separated iff Γ separates E.

Example 4:
Let E be a toplogical VS. Then the collection of subsets A of E which are absorbed by each neighborhood of 0 is a vector bornology on E. It is called the **von Neumann bornology** B of E.

If E is locally convex, then so is B.

Example 5:
Let E be a topological Hausdorff space. Then the family of relatively compact subsets of E is a vector

bornology on E, with the family of closed subsets of E as a base. It is called the **compact bornology** of a topological space.

Example 6:
Let E be a topological Hausdorff space. The family of subsets of compacts disks in E is a convex bornology on E. It is called the **bornology of compact disks** of a topological space.

Example 7:
Let E, F be topological VS. By $L(E, F)$ we denote the **vector space of all continuous linear maps** of E into F.

A subset H of $L(E, f)$ is called **equicontinuous** if:
For all neighborhoods V of 0, $V \in F$, $H^{-1} := \bigcap_{u \in H} u^{-1}(V)$ is a neighborhood of 0 in E.

The family K of equicontinuous subsets of $L(E, F)$ is a vector bornology on $L(E, F)$. K is called the **equicontinuous bornology** of $L(E, F)$, and it is convex if F is locally convex.

Since every element of $L(E, F)$ is continuous, K covers $L(E, F)$ and is hereditary under inclusion and finite union.

Let $\mathfrak{V}$ be a base of circled neighborhoods in F. Then $\forall V \in \mathfrak{V} : \exists W \in \mathfrak{V} | W + W \subset V$
Let be $H_1, H_2 \in K$, then $H_1^{-1}(W)$ and $H_2^{-1}(W)$ are neighborhoods of zero in E.
$H_1^{-1}(W) \cap H_2^{-1}(W) \subset (H_1 + H_2)^{-1}(V)$, so $(H_1 + H_2)^{-1}(V)$ is an open neighborhood of 0 in E.
Since V is an arbitrary element of a base of F, K is stable under vector addition.

K is also stable under homothetic transformations since
$(\lambda H)^{-1}(V) = \frac{1}{\lambda} H^{-1}(V)$

If H_1 is the circled hull of H, it is $H^{-1}(V) \subset H_1^{-1}(V)$, so K is stable under formation of circled hulls.

Let F be locally convex, and $\mathfrak{V}$ be a disked convex base of F.
Let be $x \in H^{-1}(V \in \mathfrak{V})$. Then $\sum_i \lambda_i h_i(x) \in V$, since V is convex. It follows that $H^{-1}(V) \subset (\Gamma H)^{-1}(V)$, so K is convex.

Example 8:
let X be a set, σ a family of subsets of X, and (F, B) a bornological set.
A family C of maps of X into F is called **σ-bounded**, if $C(A)$ is bounded in (F, B) for every $A \in \sigma$.

Let H be a subset of all maps of X into F. If all points in H are σ-bounded, the σ-bounded subsets of H define a bornology on H called the **σ-bornology**.
If (X, σ) is a bornological set, this is called the **natural bornology** on H.
A subset of H which is bounded in respect to the natural bornology is called **equibounded**.

Chapter 2

Topology-Bornology: Internal Duality

2.1 Compativle Topologies and Bornologies

2.1.1

Let E be a VS. Then let B be a bornology on E and let J be a vector topology on E.

B and J are called **compatible** if B is finer than the von Neumann bornology of (E, J).

2.1.2

A subspace of a bornological VS (E, B) is called a **bornivorous subset** if it absorbs every bounded set of E.

Let E be a convex bornological space and let V be the family of all bornivorous disks in E.

We are going to show that V is a base for the finest locally convex topology J on E compatible with the bornology of of E.

The members of V are by definition absorbent, convex and circled. V is clearly stable under homothetic transformation and finite intersections, so V is the base of a locally convex topology on E. Every bounded subset of E is bounded in the von Neumann bornology of (E, J).

If J' is a locally convex topology on E which is compatible with B, it has a base of bornivorous disks and so J is finer than J'.

The topology J is called the **locally convex topology associated with the bornology** B **of** E.
E, endowed with that topology is denoted by $\mathbb{T}E$.

2.1.3

Let (E, J) be a locally convex space. Then by definition the von Neumann bornology of (E, J) is the coarsest convex bornology on E compatible with J.

E, endowed with the von Neumann bornology, will be denoted by $\mathbb{B}E$.

2.1.4

The bornology of $\mathbb{BT}E$ is always coarser than the bornology of E. It is called the **weak bornology** of E.

Proposition:
A bornology on E and its weak bornology are equal iff the bornology of E is a von Neumann bornology of a locally convex space.

The necessity is obvious, but to prove the sufficiency, we first need to prove the following lemma:

Lemma:
For every locally convex space F, it is $\mathbb{B}F = \mathbb{BTB}F$

Proof:
Since $id : \mathbb{TB}F \to F$ is continous, $id : \mathbb{BTB}F \to \mathbb{B}F$ is bounded.

Conversely, if V is a bounded subset of of $\mathbb{B}F$, it is absorbed by each neighborhood of zero of $\mathbb{TB}F$, and therefore is bounded in $\mathbb{BTB}F$

The lemma proofs the proposition.

A convex bornology on E is called a **topological bornology** if $E = \mathbb{BT}E$, i.e. if it is the von Neumann bornology of a locally convex space.

2.1.5

Let (E, J) be a locally convex space. Then the topology of $\mathbb{TB}E$ is always finer than J.

Proposition:
Let E be a locally convex space. Then $E = \mathbb{TB}E$ iff the topology of E is the locally convex topology associated with a convex bornology on E.

Proof:
The nessecity is obvious. To proove the sufficiency, we first proof the following Lemma:

Lemma:
For each convex bornological space F, it is $\mathbb{T}F = \mathbb{TBT}F$

Proof:
Since $id : F \to \mathbb{BT}F$ is bounded, $id : \mathbb{T}F \to \mathbb{TBT}F$ is continuous.

Conversely, let V be a neighborhood of 0 in $\mathbb{T}F$, then it is a union of bornivorous disks. Since the bounded subsets in $\mathbb{BT}F$ are those sets absorbed by each neighborhood of zero in $\mathbb{T}F$, V is a union of bornivorous discs in respect to $\mathbb{BT}F$ in is therefore open in $\mathbb{TBT}F$.

The Lemma proofs the proposition.

Let E be a locally convex space. The topology of E is called a **bornological topology**, if $E = \mathbb{TB}E$.

Proposition:
Every metrizable locally convex topology E is bornological.

Proof:
We have to show that $E = \mathbb{TB}E$. Since $\mathbb{TB}E$ is always finer than E, it suffices to show that $id : E \to \mathbb{TB}E$ is continuous.
This equivalent to show that every bornivorous disk of $\mathbb{B}E$ is a neighborhood of zero in E.

Such a disk absorbs all sequences which converge to 0, and therefore is a neighborhood in E, by the following Lemma:

Lemma:
In a metrizable topological VS E, every circled set which absorbs all sequences converging to 0 is a neighborhood of 0.

2.2 Characterisation of Bornological Topologies

2.2.1

Let E, F be locally convex spaces, and let u be a linear map $u : E \to F$.

If u is continuous, it is bounded in respect to $\mathbb{B}E$, $\mathbb{B}F$. The converse is not true in general.

2.2.2

We will show that the locally convex topologies for which each bounded linear map into a LCS is continuous are exactly the bornological topologies.

Proof:
Let E be a LCS.

Assume E to be a bornological LCS and u to be a linear map $u : E \to F$.
The for each disked neighborhood V of zero in F, $u^{-1}(V)$ is abornivorous disk in E, and therefore a neighborhood of zero in E, since $E = \mathbb{TB}E$.

Suppose every bounded (i.r.t. the von Neumann bornology) linear map $u : E \to F$ is continuous. Since the identity $id : \mathbb{B}E \to \mathbb{BTB}E$ is bounded, and therefore $id : E \to \mathbb{TB}E$ is continuous. This leads to the topological identity $\mathbb{TB}E = E$.