Quantum field theory as a new way to quantum gravity

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University of Rome "Tor Vergata"

Symposium "Raum und Materie", 17.10.2012

[1] based on the joint work with prof. Klaus Fredenhagen

Outline of the talk

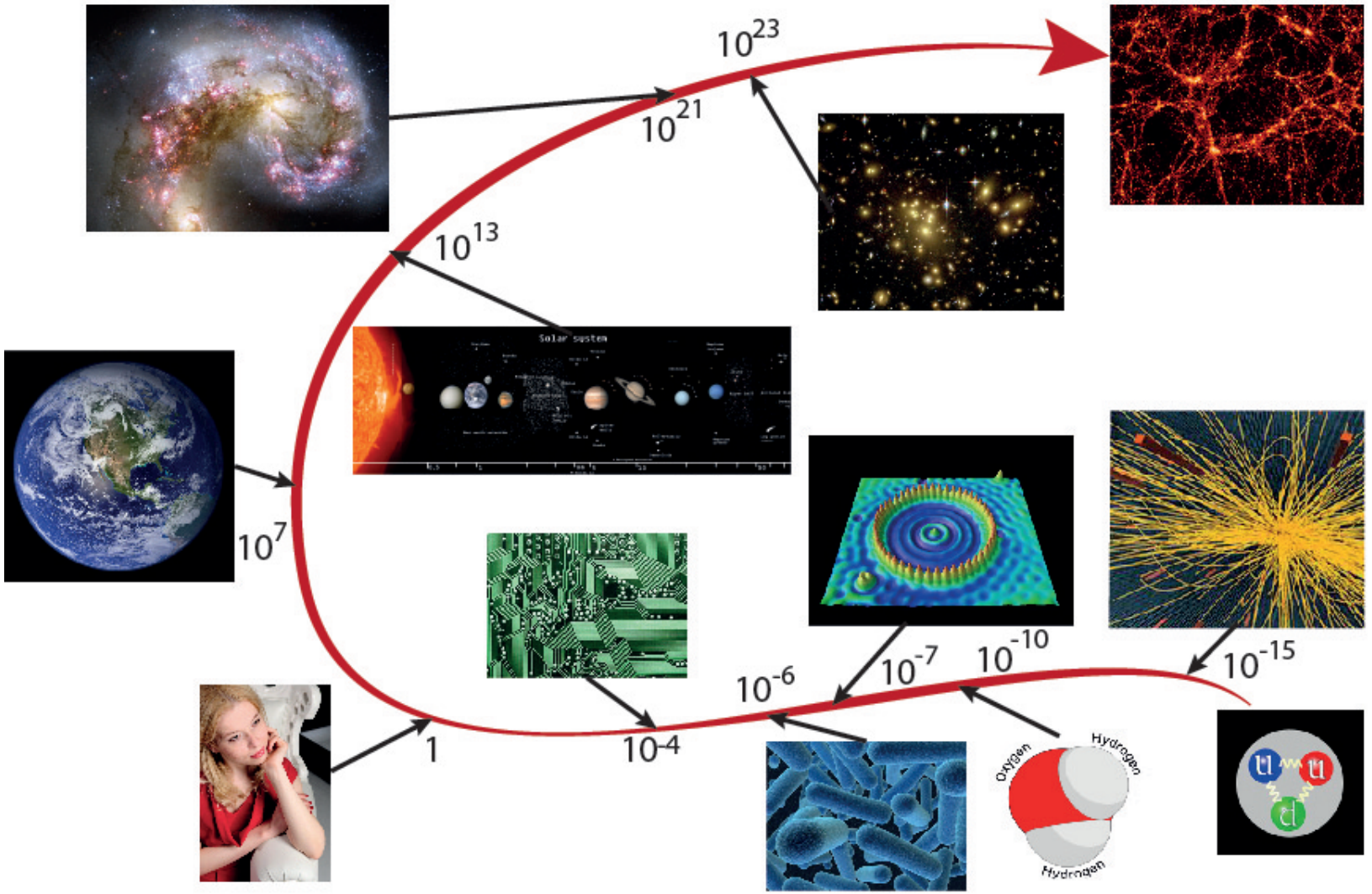
1. Why do we want to quantize gravity?
2. QFT on curved spacetime as a way to QG
3. Symmetries in physics

1 Why do we want to quantize gravity?

2 QFT on curved spacetime as a way to QG

3 Symmetries in physics

Scales in the Universe



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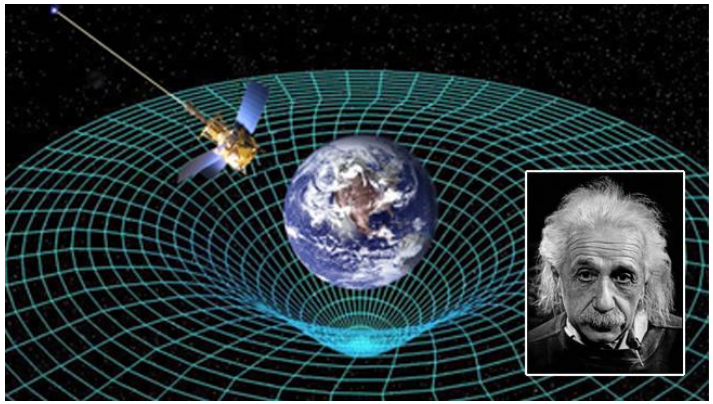

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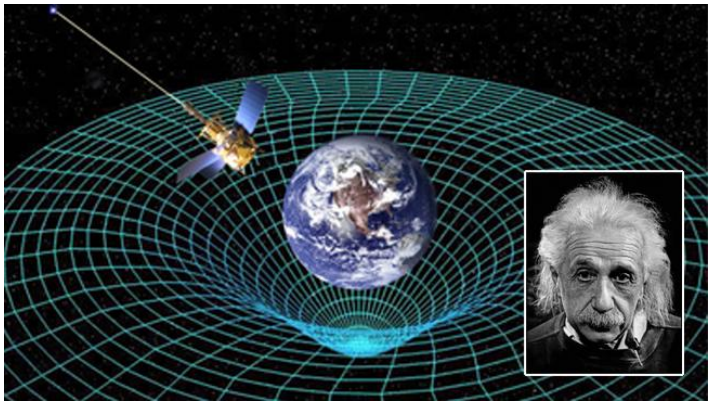

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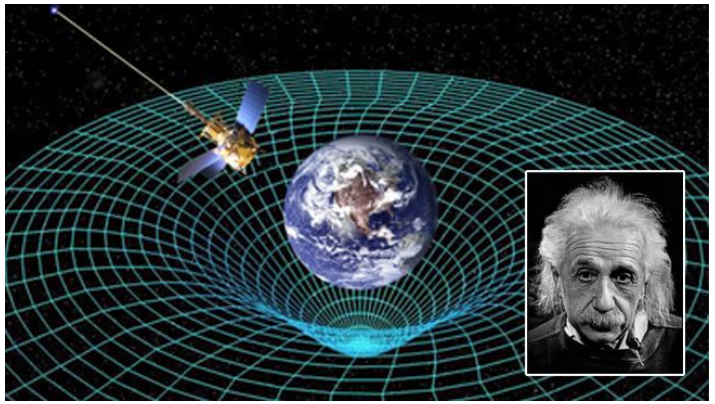

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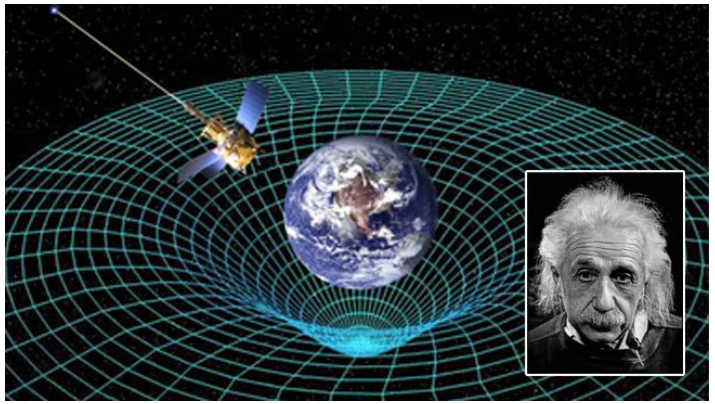
- In this theory space and time are combined to one structure, called spacetime.
- In this theory the gravitational "force" arises as a consequence of the spacetime curvature.
- This means that the spacetime *itself* is dynamical and subject to certain equations (Einstein's equations).



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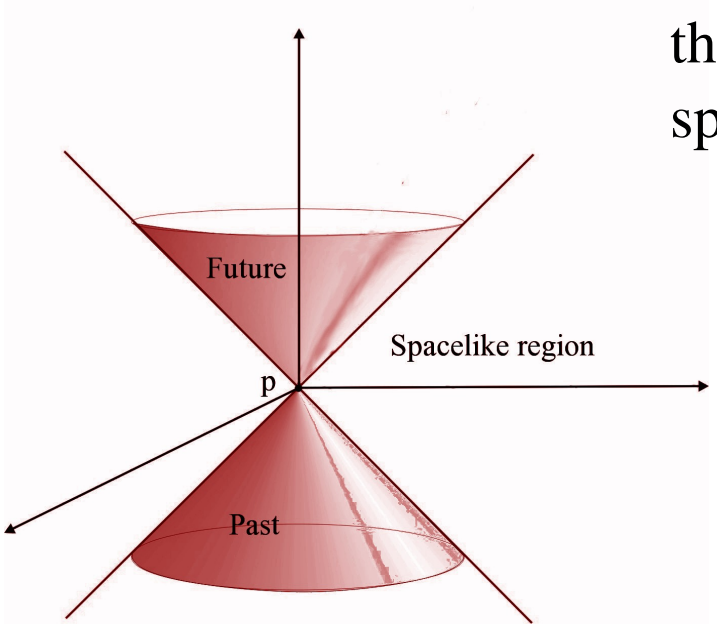

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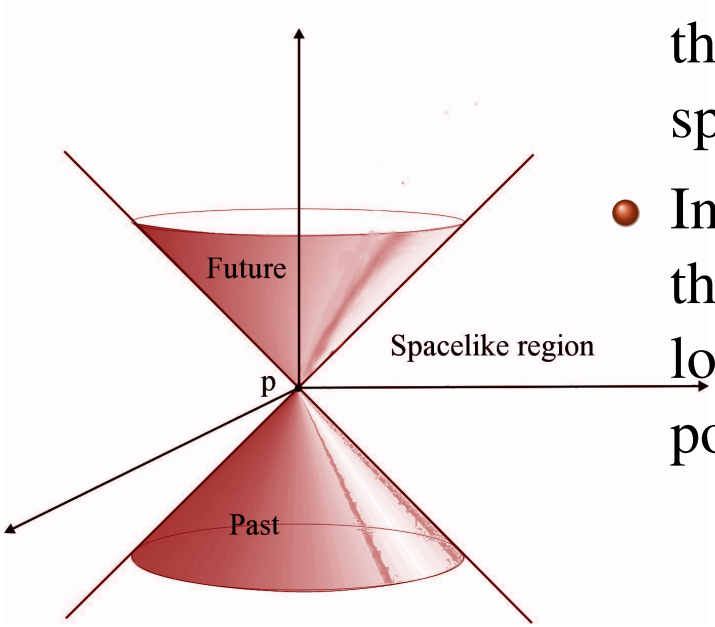

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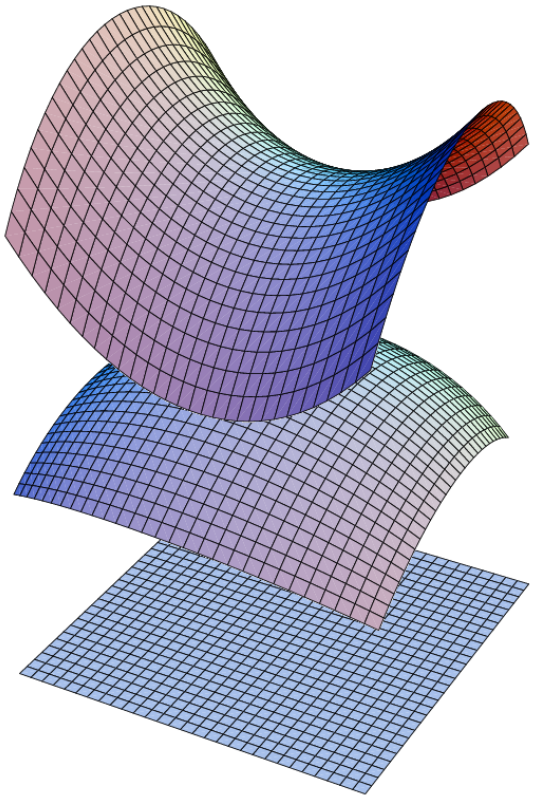

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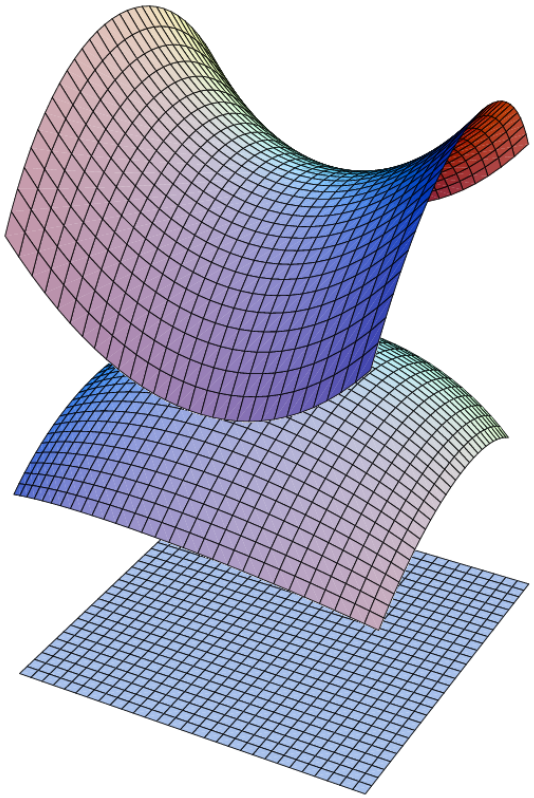

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- The principle of covariance tells us that physical laws have to be independent of the choice of spacetime coordinates

How matter influences the spacetime

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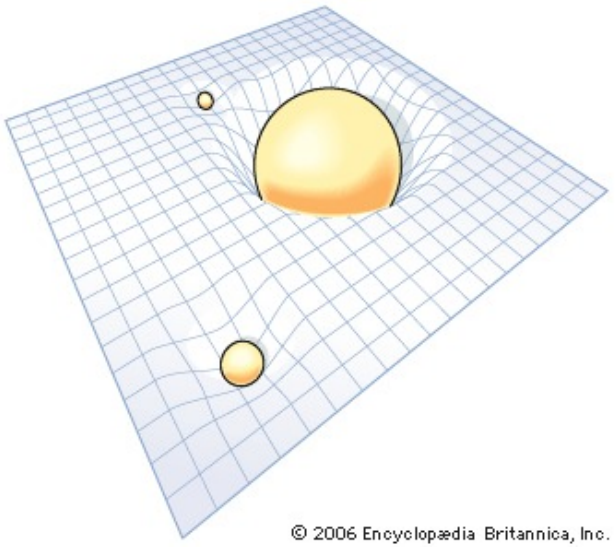

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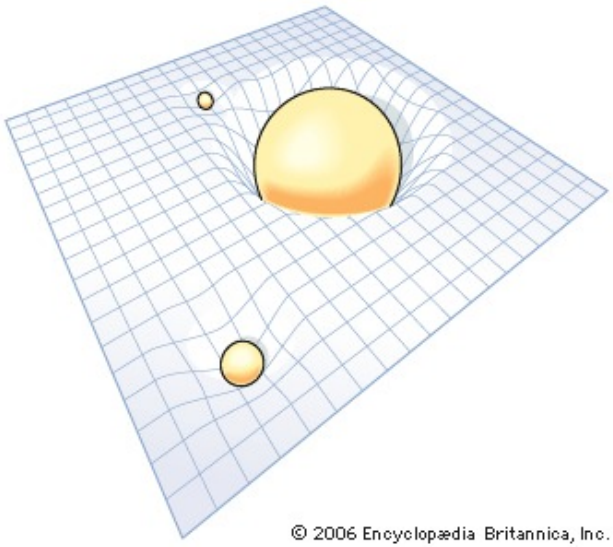

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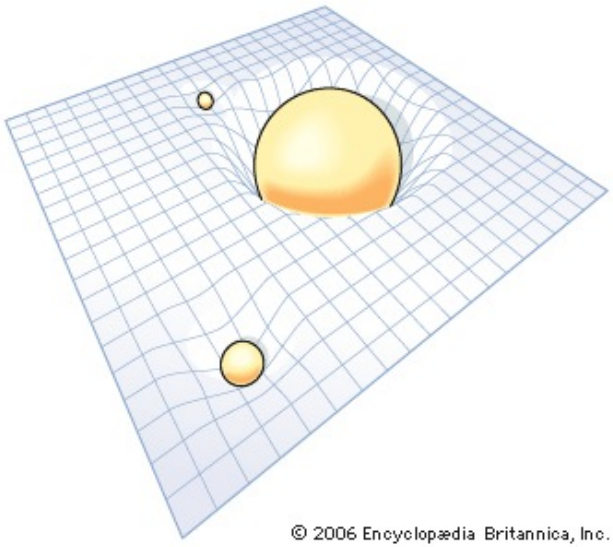

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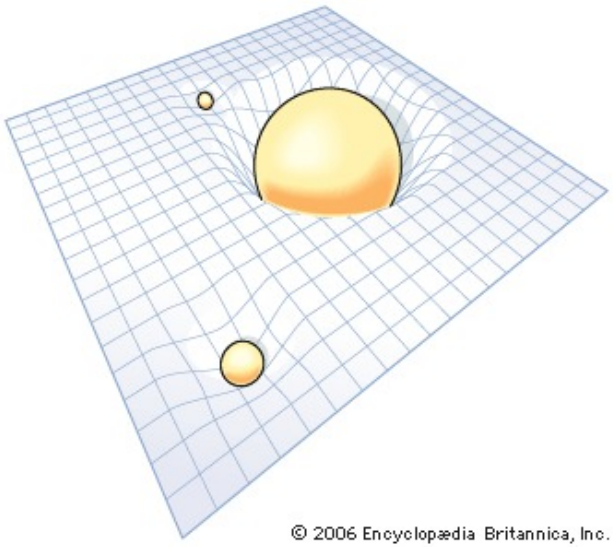

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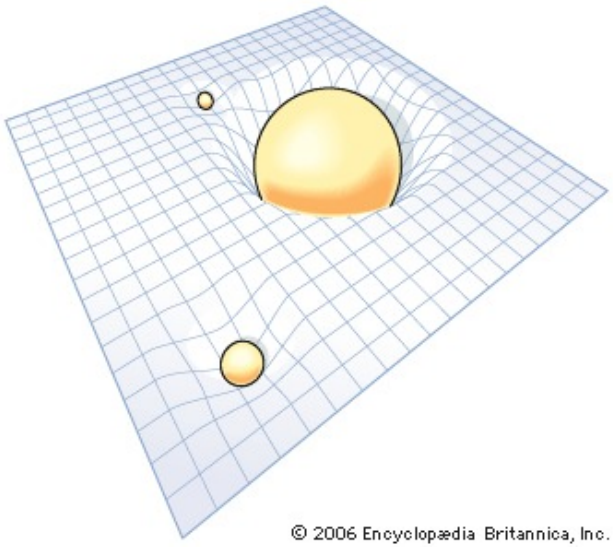
- From this equation it is clear that the change of spacetime influances the matter, but also change of the matter influences the curvature of spacetime!



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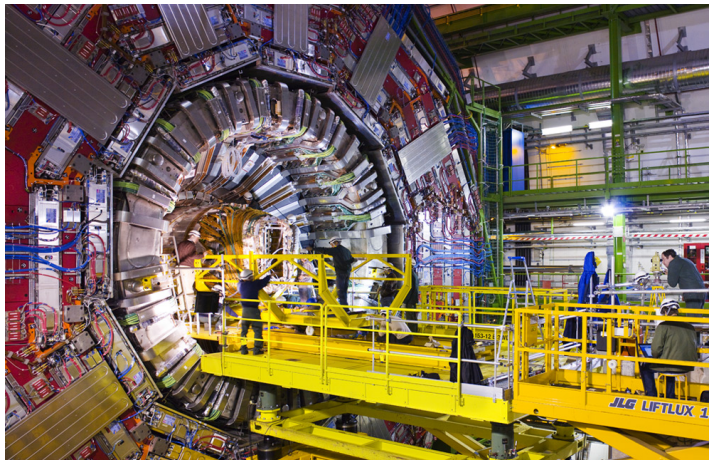

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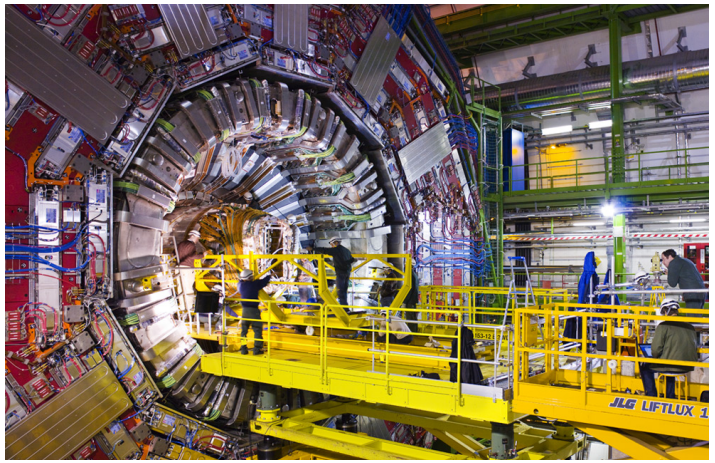

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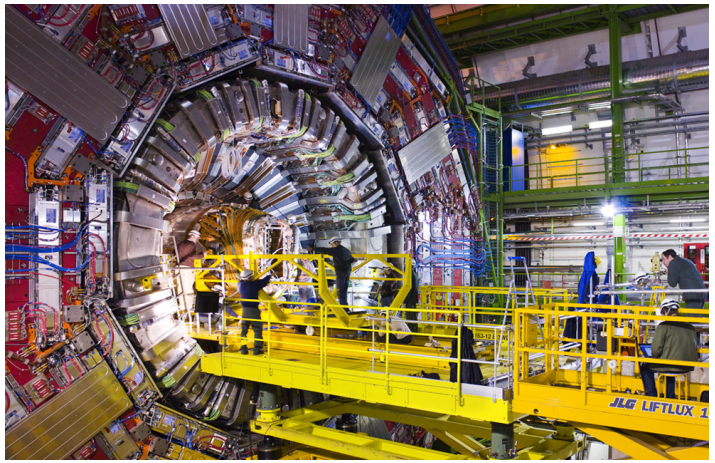

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- Experiments with particle collisions (for example at CERN) can be understood with the use of the scattering theory,
- In QFT spacetime is *fixed*, it has no dynamics.

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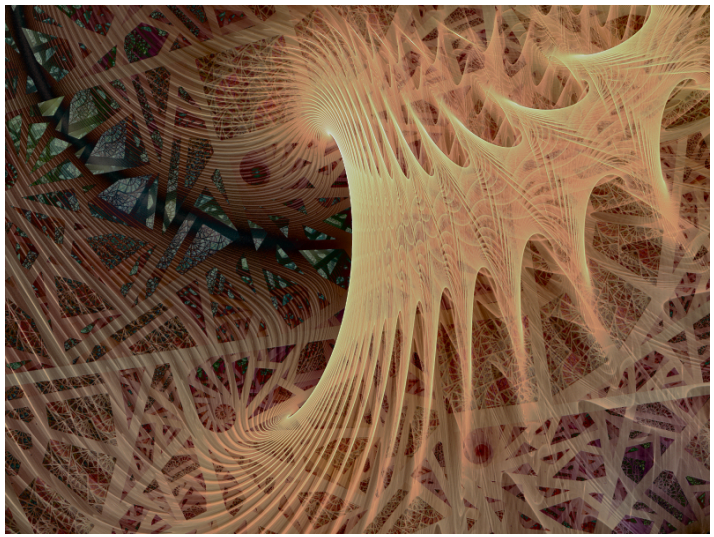
- Such situations can be relevant for the physics of black holes or the early Universe.

Quantum gravity

- Quantum gravity has to unify GR and QFT.



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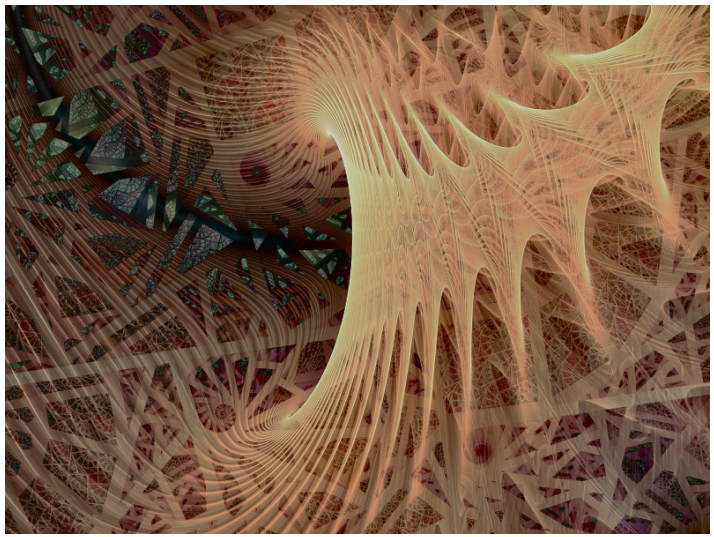

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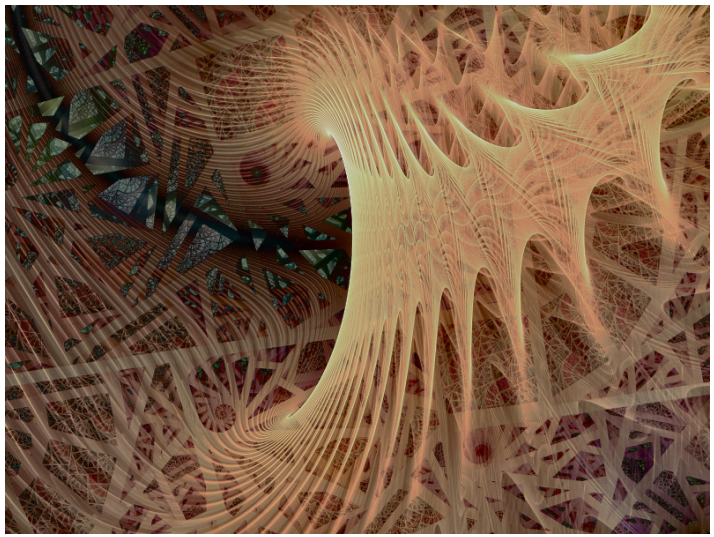

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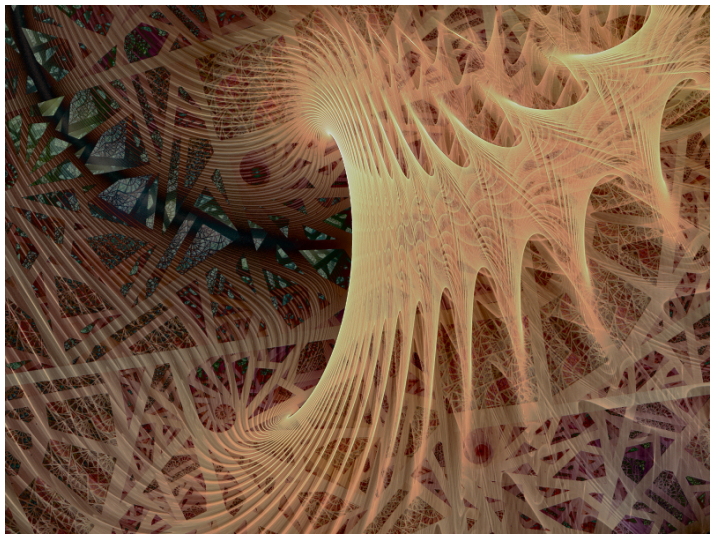
- What does it mean for geometry to be quantized?



Possible approaches

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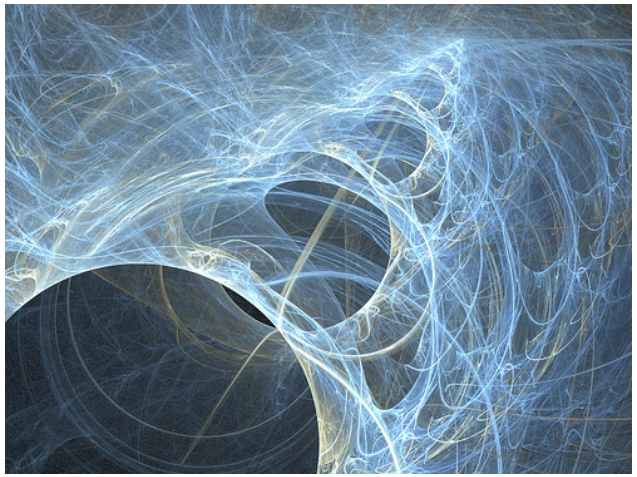

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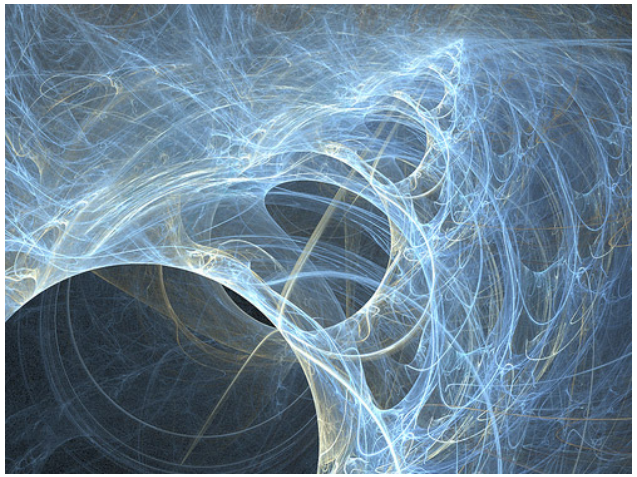

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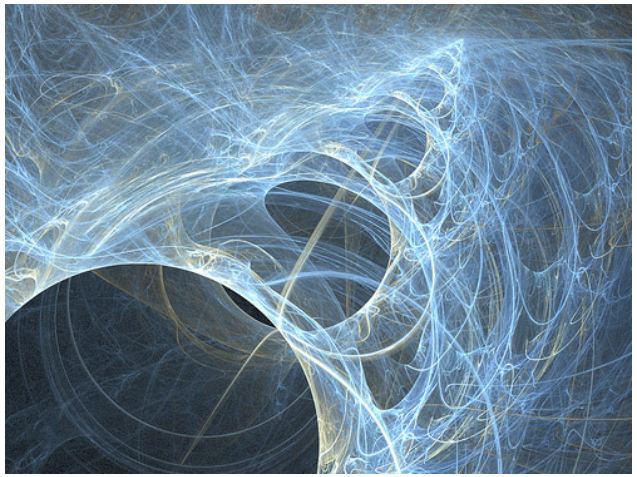

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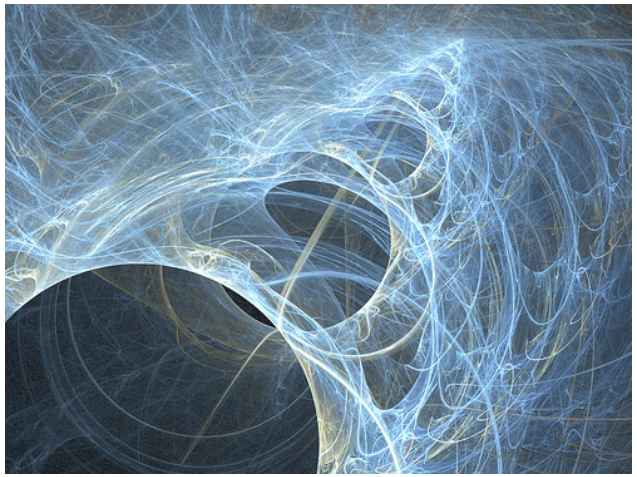
- There are also other, to some extend related, approaches like dynamical triangualtions or spin foam models.
- The approach which we propose is somewhat more conservative: let's start from the side of QFT and see how much can we learn about QG from this point of view.



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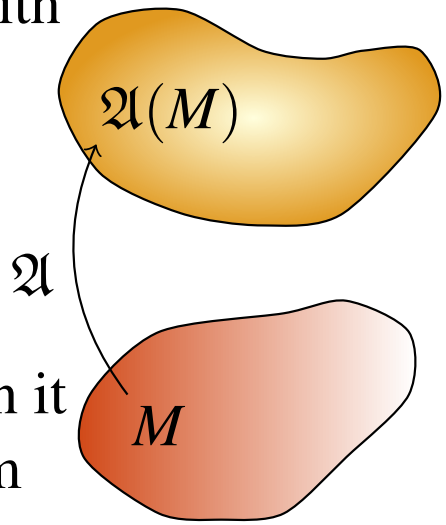

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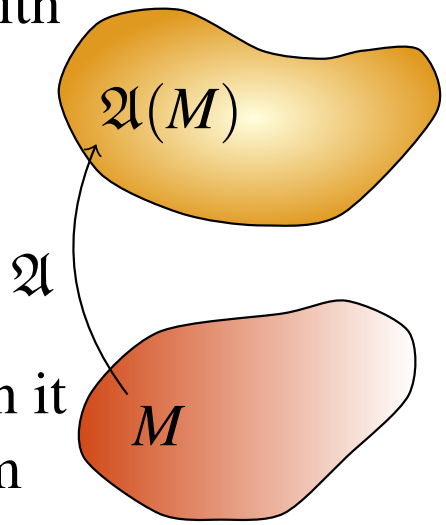

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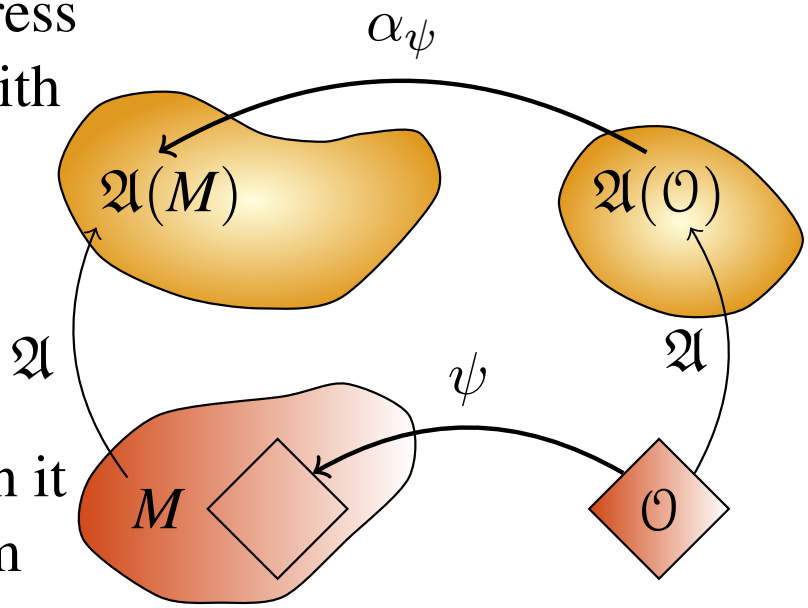
- With bounded regions of spacetime one associates local algebras of observables.
- The principle of covariance known from GR is realized by imposing the so called general local covariance.



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- The covariance property reads:

$$\alpha_{\psi'} \circ \alpha_\psi = \alpha_{\psi' \circ \psi}\,, \qquad \alpha_{\mathrm{id}_M} = \mathrm{id}_{\mathfrak{A}(M)}\,,$$

Further axioms

One can also include two further axioms which are important in QFT: causality and time-slice axiom.

- **Causality**: If there exist admissible embeddings $\psi_j : M_j \to M, j = 1, 2$, such that the sets $\psi_1(M_1)$ and $\psi_2(M_2)$ are causally separated in M, then:

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- **Time-slice axiom:** If the morphism $\psi : M \to M'$ is such that $\psi(M)$ contains a Cauchy-surface in M', then α_ψ is an isomorphism (Remark: Cauchy surface= every inextendible causal curve intersects it only once).

QFT on curved spacetimes (applications)

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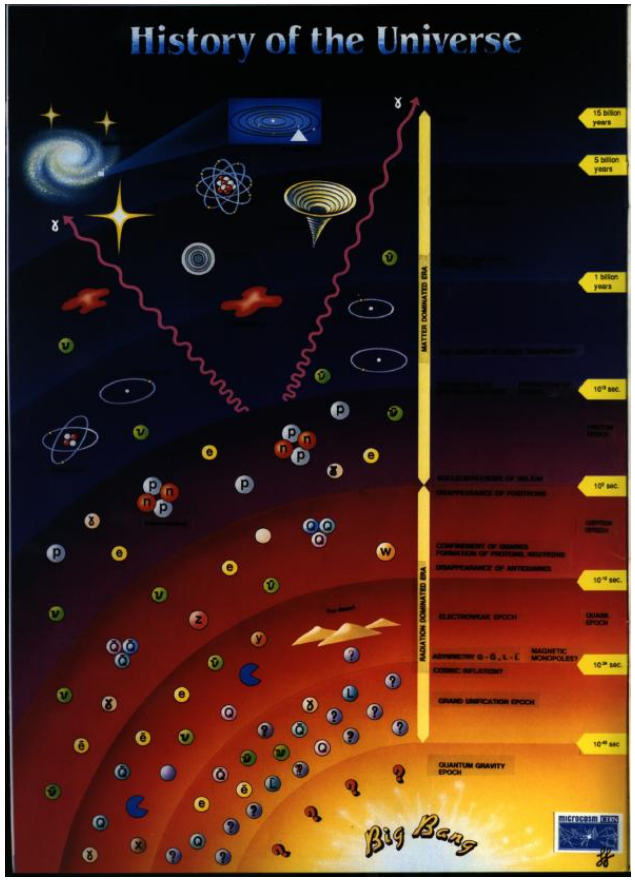

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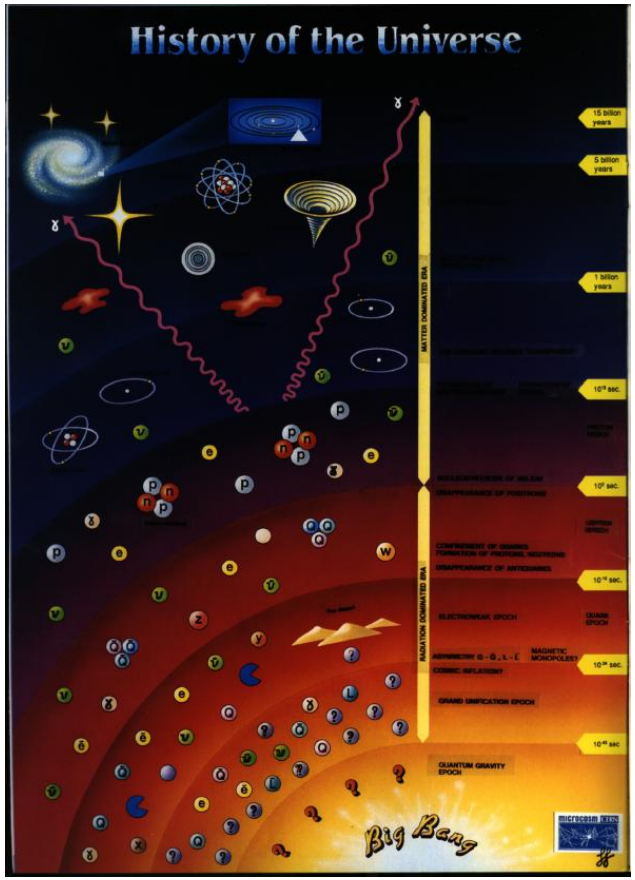

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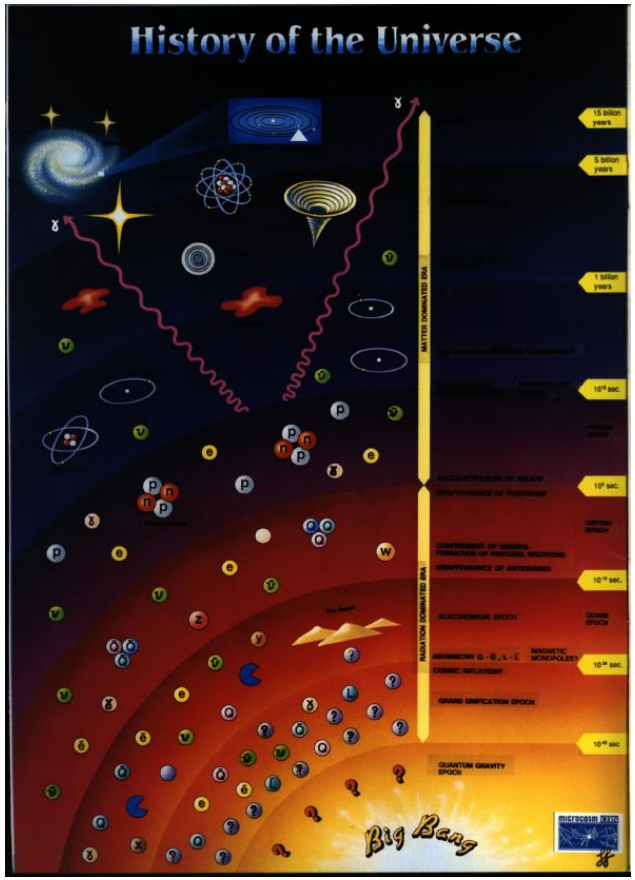

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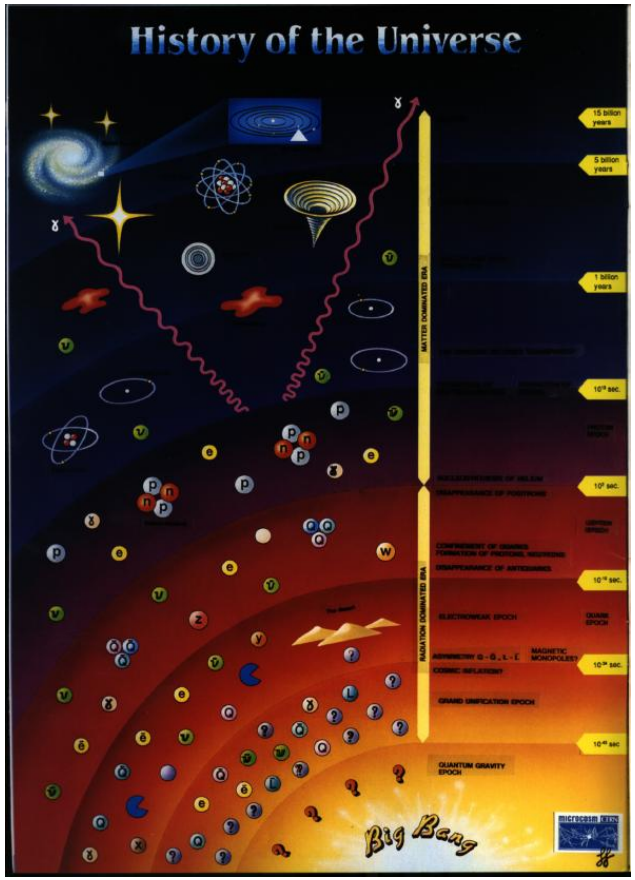
- This has applications in cosmology: the background spacetime is homogenous and isotropic and we model the evolution of the universe by studying the behavior of quantum and classical matter (dust) put into it.



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- The renormalizability problem is solved by using the so called Epstein Glaser renormalization
- In my Ph.D. project I approached 2 of the other problems: the reparametrization (diffeomorphism) invariance and the problem with observables.

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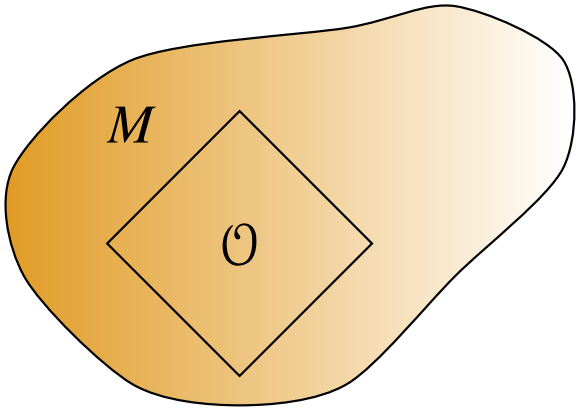

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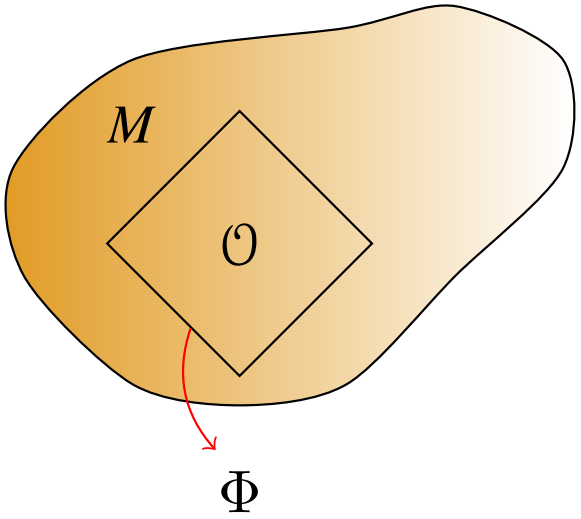

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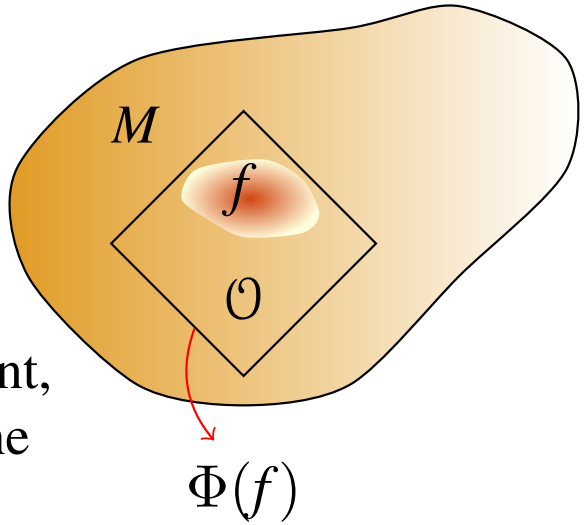

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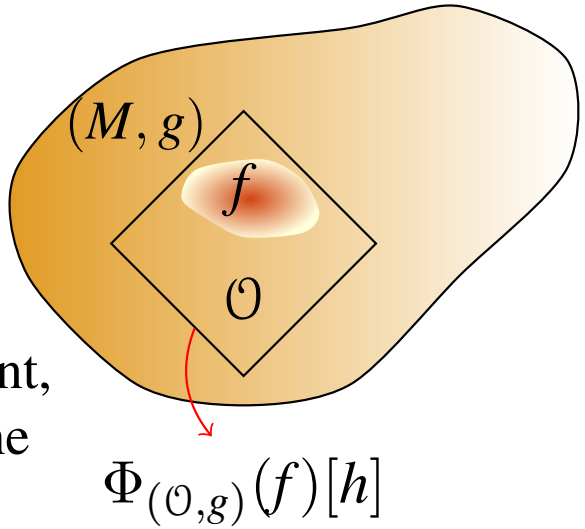

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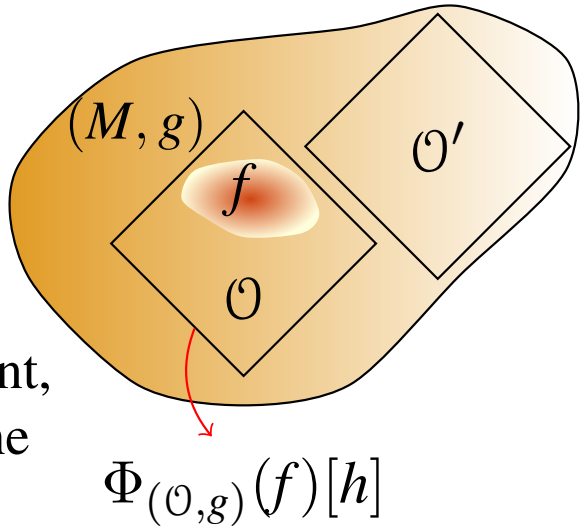

- We can think of the measured observable as a perturbation of the fixed background metric: a tentative split $\tilde{g}_{\mu\nu} = g_{\mu\nu} + h_{\mu\nu}$.
- Diffeomorphism transformation: move our experimental setup to a different region $\mathcal{O}'$.

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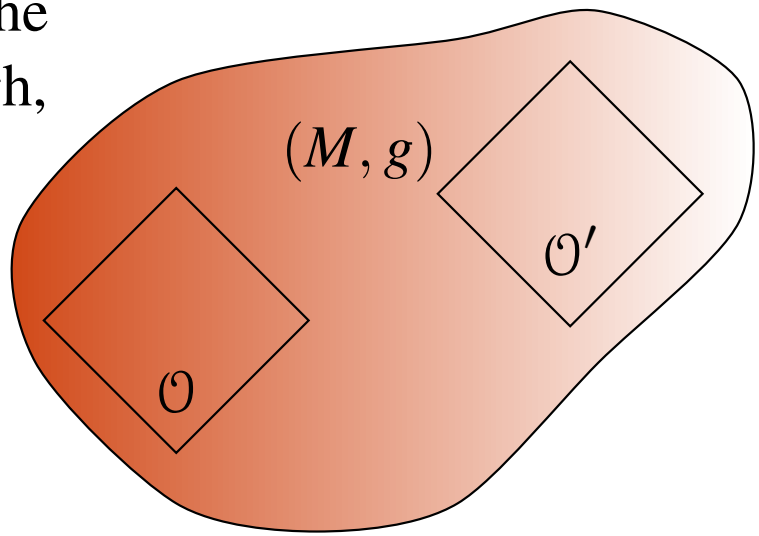

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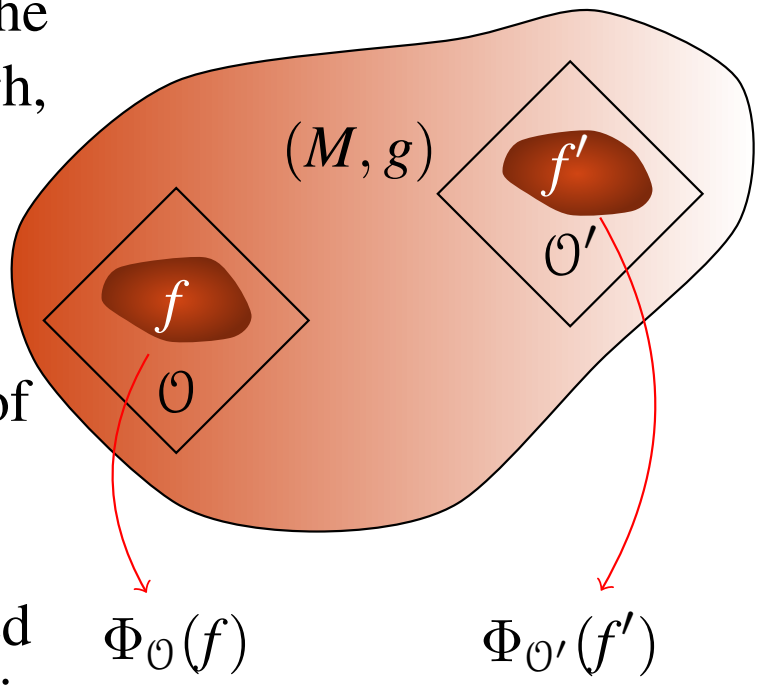

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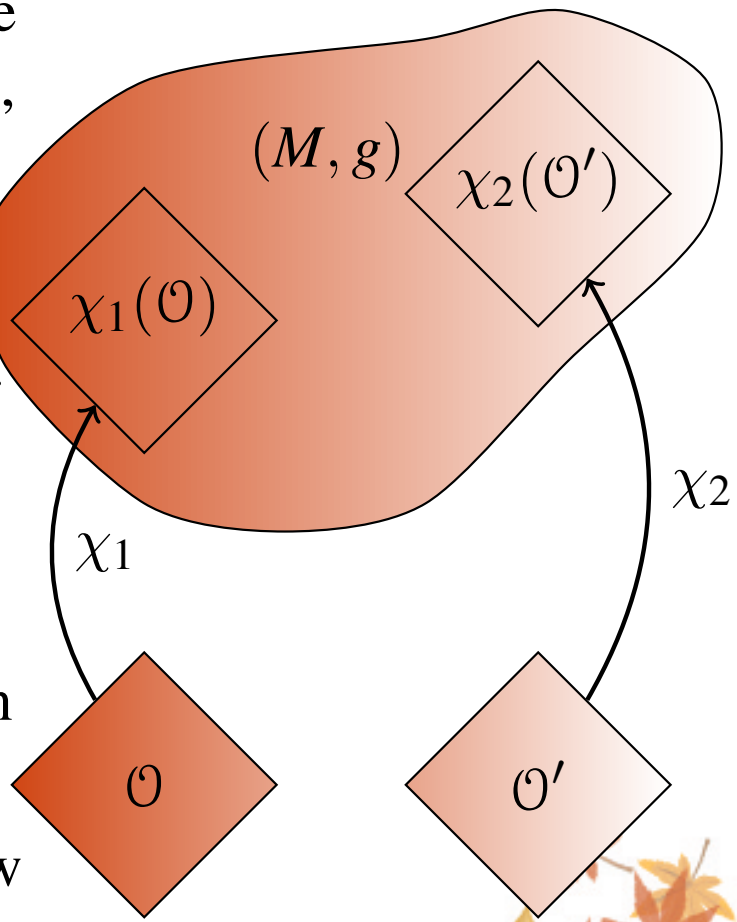
- Physical picture: a field tells us how to identify observables living in different regions.



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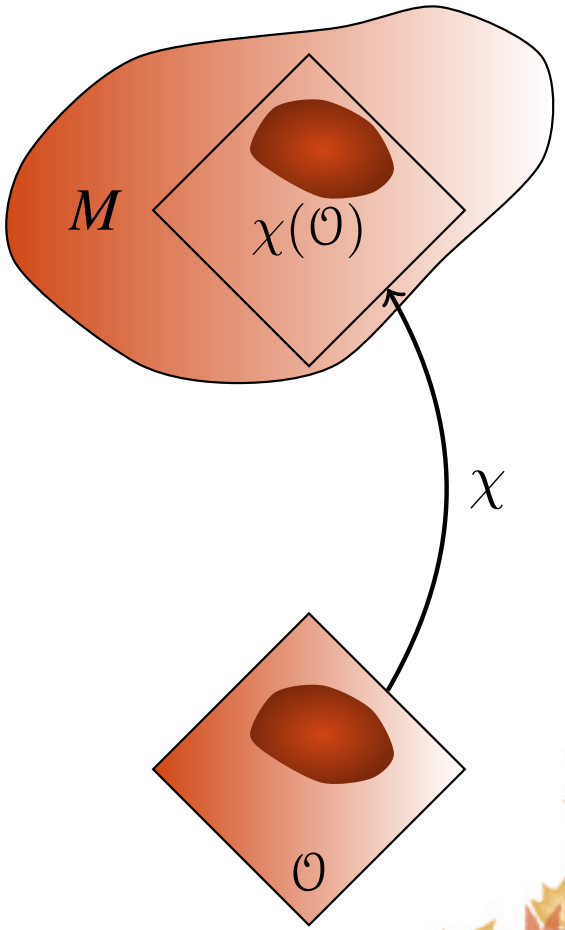

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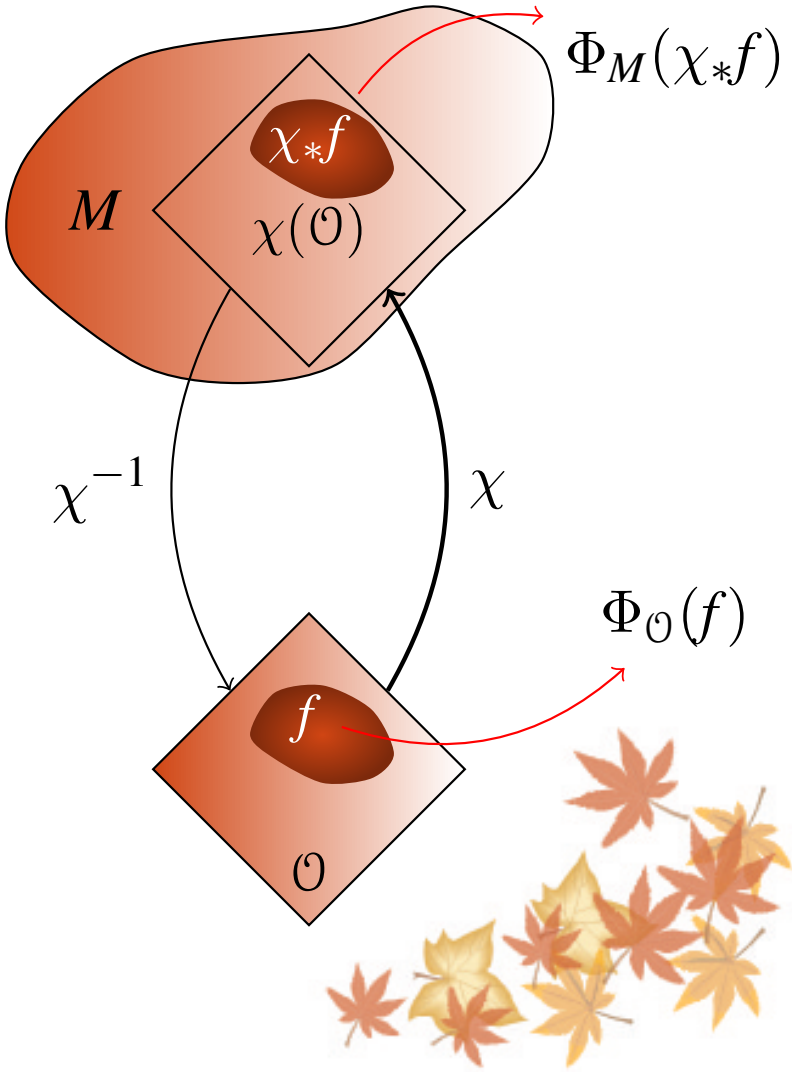

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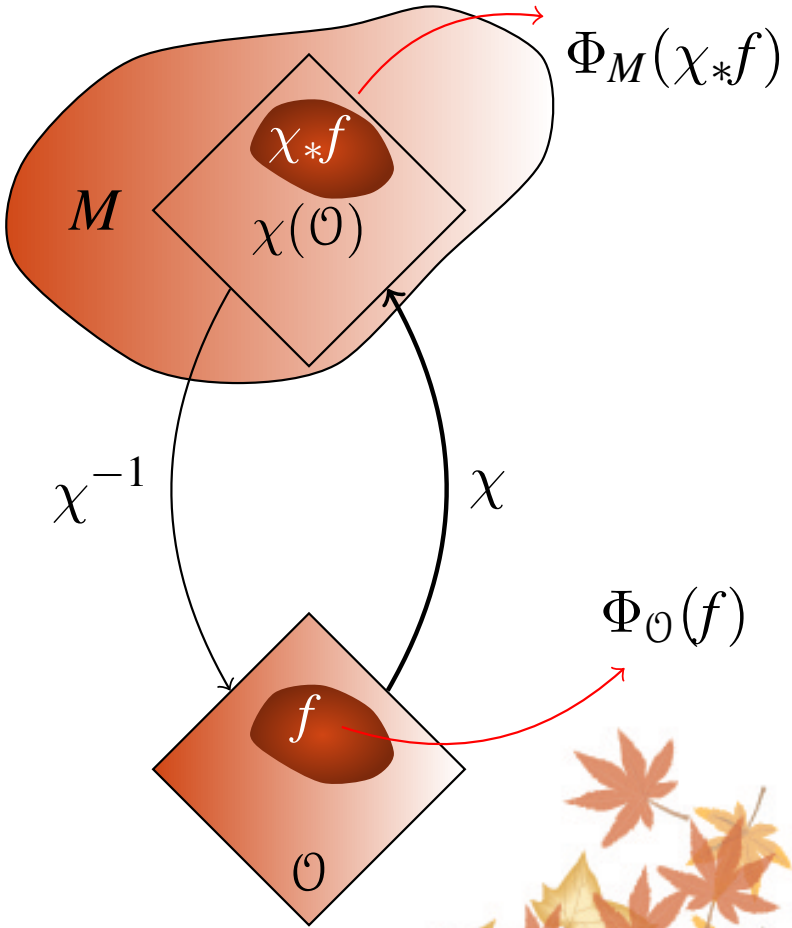
- But this is not enough. We need not only diffeomorphism covariance, but invariance



1 Why do we want to quantize gravity?

2 QFT on curved spacetime as a way to QG

3 Symmetries in physics

Mathematical description of physical systems

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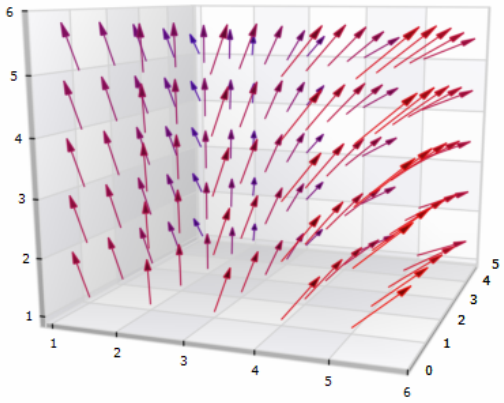

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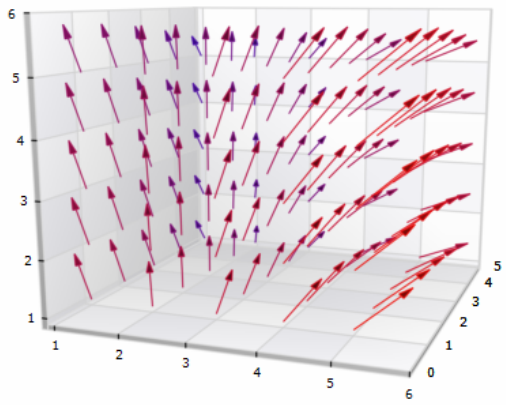

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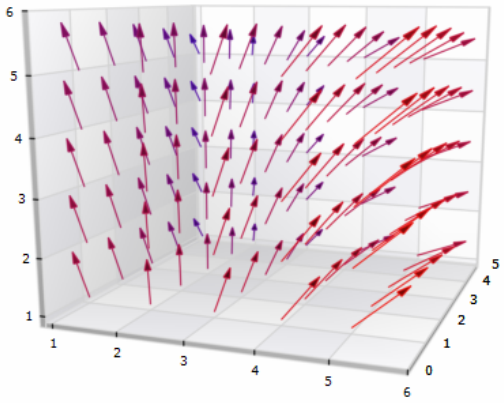

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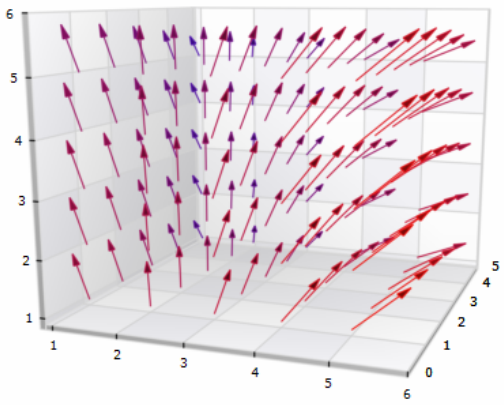

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- The choice of variables can be in some sense redundant. In this case we speak of symmetries.

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- We say that the above transformations are symmetries of S_{EH}. It follows that if g is a solution of the field equation, then so is g'.

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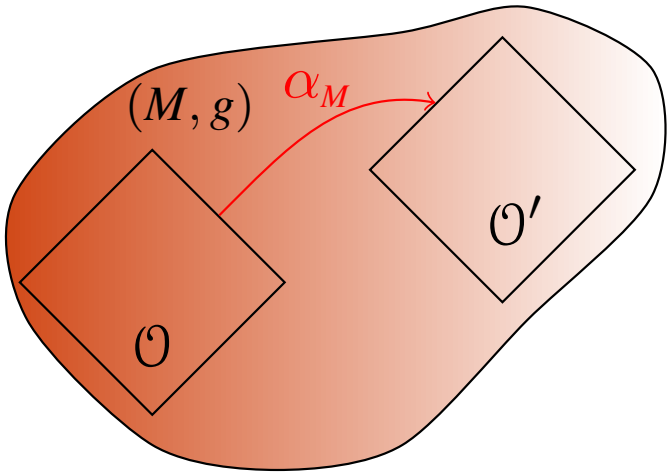

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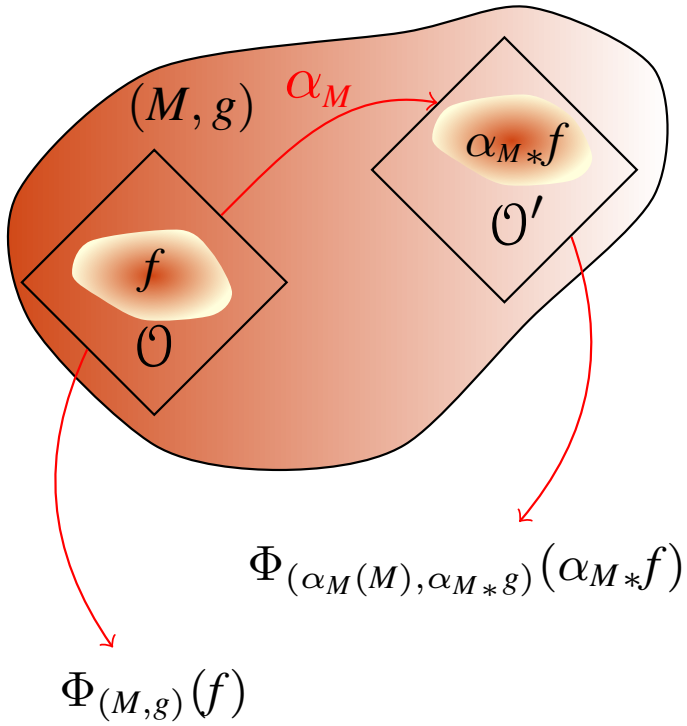

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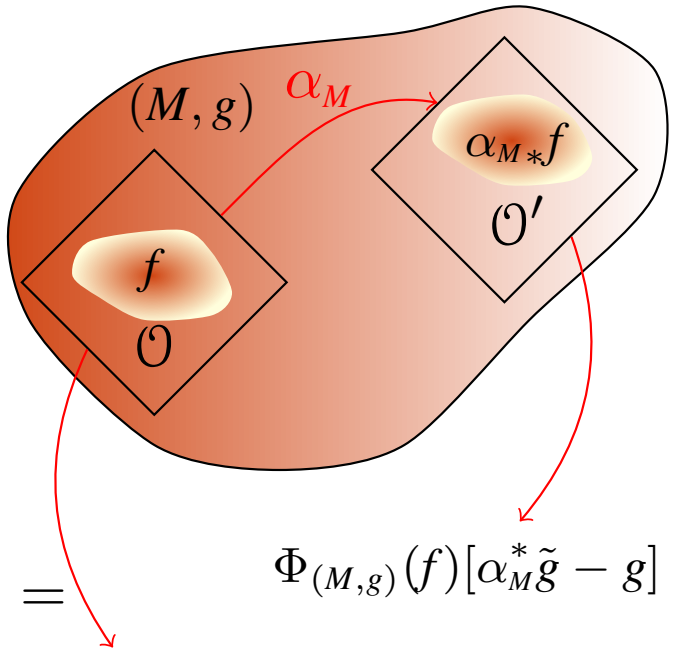

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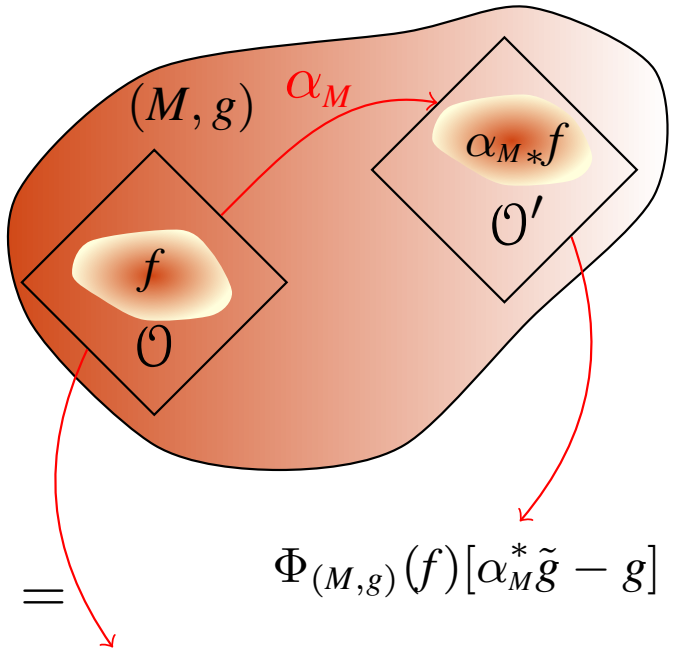

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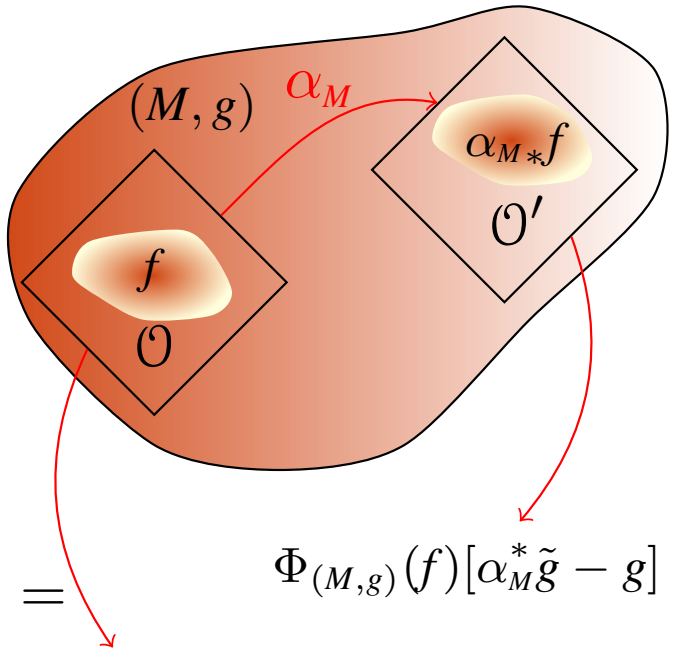

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- Example: $\int R[\tilde{g}] f \, \mathrm{d}\,\mathrm{vol}_{(M,\tilde{g})}$ is invariant under the action of $\mathfrak{X}$, but $\int R[\tilde{g}] f \, \mathrm{d}\,\mathrm{vol}_{(M,g)}$ is not.

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- The field equations are the Maxwell's equations:

$$\begin{aligned} \nabla_{[\mu} F_{\nu\alpha]} &= 0 \\ \nabla_\mu F^{\mu\nu} &= 4\pi J^\nu, \end{aligned}$$

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- Solutions of this equation have a local symmetry: $A'_\mu = A_\mu + \nabla_\mu \chi$ for an arbitrary function $\chi(x)$.

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where $*$ is the Hodge dual, $F := \mathrm{d}A + A \wedge A$, $A \in \Omega^1(M) \otimes g$ is a Lie algebra valued 1-form, and g is the Lie algebra of a group G (the symmetry group). For QED the symmetry group is $U(1)$. In coordinates one writes:

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- The gauge symmetry takes the form: $\tilde{A} = A + d\xi + [\xi, A]$, $\xi \in \mathfrak{g} \doteq \mathcal{C}^\infty(M, g)$.

Yang-Mills theories

- Yang-Mills theories are used in particle physics to describe the fundamental forces of nature,
- The YM action S_{YM} reads:

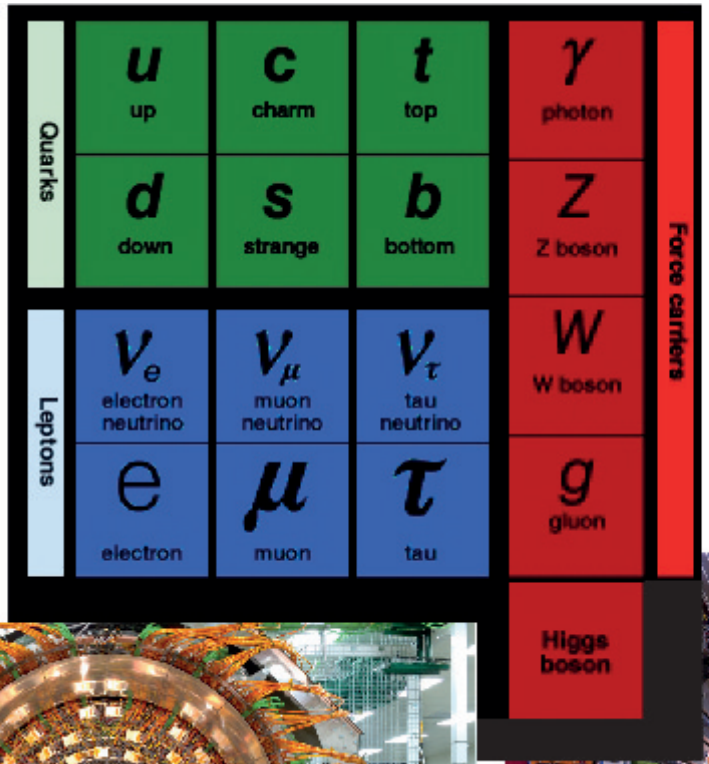
$$S_{\mathrm{YM}} = \frac{1}{4}\int \mathrm{Tr}\left[*F \wedge F\right] ,$$

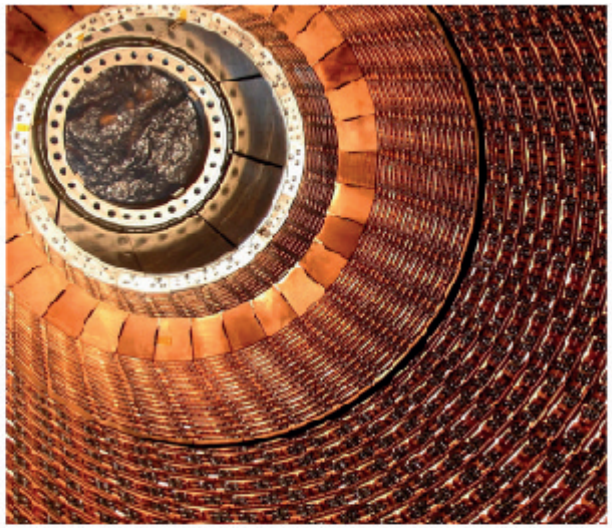
where $*$ is the Hodge dual, $F := \mathrm{d}A + A \wedge A$, $A \in \Omega^1(M) \otimes g$ is a Lie algebra valued 1-form, and g is the Lie algebra of a group G (the symmetry group). For QED the symmetry group is $U(1)$. In coordinates one writes:

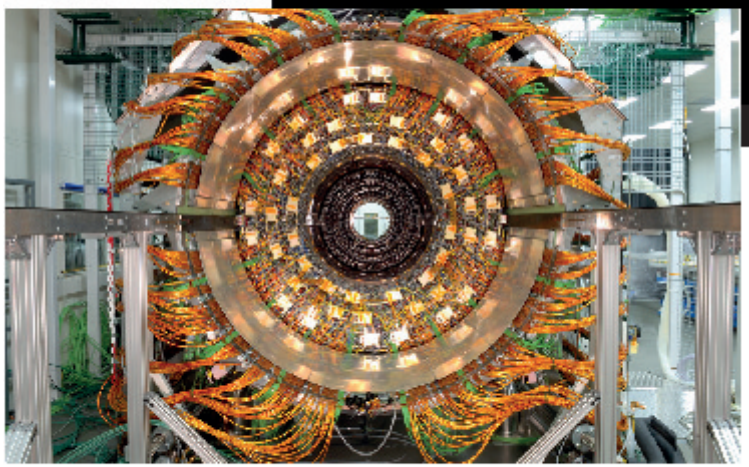
$$F^a_{\mu\nu} = \nabla_\mu A^a_\nu - \nabla_\nu A^a_\mu + f^a_{bc} A^b_\mu A^c_\nu .$$

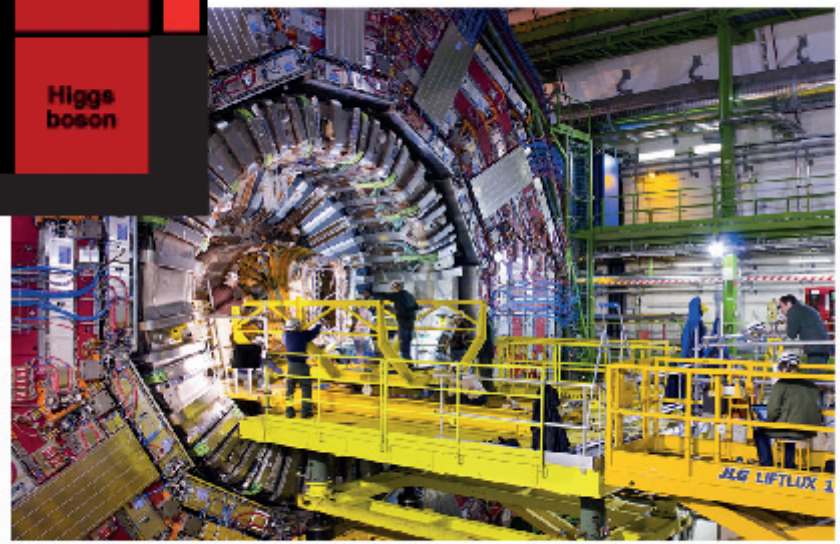
- The gauge symmetry takes the form: $\tilde{A} = A + d\xi + [\xi, A]$, $\xi \in \mathfrak{g} \doteq \mathcal{C}^\infty(M, g)$.
- In particle physics one uses the Standard Model, which is defined by the symmetry group: $G = U(1) \times SU(2) \times SU(3)$.

Standard Model









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- In particular we proposed a mathematically consistent definition of diffeomorphism invariant observables in GR.

Thank you for your attention!